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2004 J. Phys. A: Math. Gen. 37 1241

(http://iopscience.iop.org/0305-4470/37/4/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 1241–1250 PII: S0305-4470(04)66993-9

On a system of differential equations with fractional derivatives arising in rod theory

Teodor M Atanackovic[1] **and Bogoljub Stankovic**[2]

[1] Department of Mechanics, Faculty of Technical Sciences, POB 55, University of Novi Sad, 21121 Novi Sad, Serbia and Montenegro
[2] Department of Mathematics, University of Novi Sad, 21000, Novi Sad, Serbia and Montenegro

Received 23 July 2003
Published 9 January 2004
Online at stacks.iop.org/JPhysA/37/1241 (DOI: 10.1088/0305-4470/37/4/012)

Abstract
We study a system of equations with fractional derivatives, that arises in the analysis of the lateral motion of an elastic column fixed at one end and loaded by a concentrated follower force at the other end. We assume that the column is positioned on a viscoelastic foundation described by a constitutive equation of fractional derivative type. The stability boundary is determined. It is shown that as in the case of an elastic (Winkler) type of foundation the stability boundary remains the same as for the column without a foundation! Thus, with the solution analysed here, the column exhibits the so-called Hermann–Smith paradox.

PACS numbers: 46.70.De, 83.60.Bc

1. Introduction

Consider a rod of length L fixed at the end A and free at the other end B (see figure 1). Let S be the arc-length of the rod axis so that $S \in [0, L]$. The rod is loaded by a concentrated follower force, at the free end B. The rod is positioned on a viscoelastic foundation of the fractional derivative type. Our goal in this paper is to formulate differential equations describing small lateral motion of the rod and to examine the stability conditions, i.e., the conditions for which the displacement of the rod remains finite.

For the case when the motion of the column is not restricted by a foundation (elastic or viscoelastic) the column shown in figure 1 becomes the well-known Beck's column. Our analysis, that is our choice of the type of foundation, is motivated by the so-called Hermann–Smith paradox. Namely, in [1] Smith and Herrman analysed the stability of a column loaded by a follower force. It was assumed that the column is positioned on a Winkler-type elastic foundation. They obtained intuitively unexpected behaviour of the critical load for flatter: the critical load was independent of the foundation modulus! The frequency of vibration of the beam increases with increasing foundation modulus, but the magnitude of the critical load is not affected. In [2] the problem was reconsidered. It was argued in [2] that, as in the

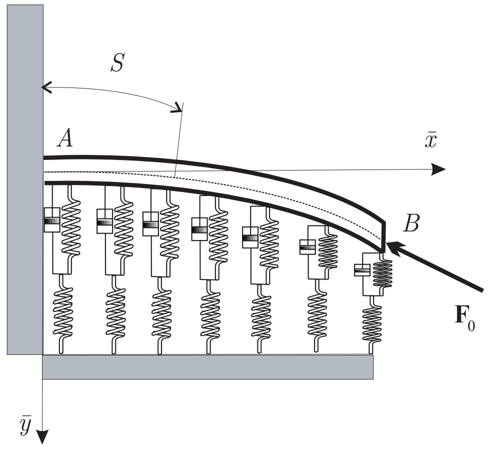

Figure 1. Coordinate system for Beck's column on a viscoelastic foundation.

situation of the so-called Ziegler [3] paradox (small internal viscosity decreases the stability boundary), here in the case of nonconservative load the paradox is: 'a consequence of uncritical application of the small-oscillation method'. In [2], the authors assumed that column is made of linearly viscoelastic material and showed that the paradox disappears, that is the critical load depends on the rigidity of the base. This led the authors of [2] to state that: 'the elastic idealization of deformable systems in nonconservative stability problems becomes physically meaningless'. Recently, a summary of results on the stability of columns subjected to follower loads has been given in [5]. The complete solution of the Hermann–Smith problem (allowing variable stiffness of the foundation) for the case of an elastic column is presented in [6].

Our analysis will show that the Hermann–Smith paradox remains if one keeps the column elastic but assumes that the foundation is viscoelastic. We shall show this for a particular type of viscoelastic foundation that is described by a fractional derivative type of constitutive equation. This model of viscoelastic foundation is used, for example, in [4] to model railpads in a study of the stability of railway track.

2. The mathematical model

Let $\bar{x}$–A–$\bar{y}$ be the fixed rectangular Cartesian coordinate system with the origin at fixed point b of the column. In what follows we consider only the plane motion of the column (the motion in the $\bar{x}$–A–$\bar{y}$-plane). Equations of motion for the column read (see [10])

$$\frac{\partial H}{\partial S} = -q_x \qquad \frac{\partial V}{\partial S} = -q_y \qquad \frac{\partial M}{\partial S} = -V\frac{\partial x}{\partial S} + H\frac{\partial y}{\partial S} - m \tag{1}$$

where x and y are coordinates of an arbitrary point on the column axis in the deformed state along the $\bar{x}$- and $\bar{y}$-axes, respectively, H and V are components of the contact force (representing the influence of the part of the column $[0, S)$ on the part $[S, l]$) along the $\bar{x}$- and $\bar{y}$-axes, respectively and q_x, q_y and m are the intensities of the distributed forces per unit length of the column axis along the $\bar{x}$- and $\bar{y}$-axes and the intensity of the distributed couple, respectively. We assume that the distributed forces come from the inertial force of the column element and from the foundation. Therefore

$$q_x = -\rho\frac{\partial^2 x}{\partial t^2} \qquad q_y = -\rho\frac{\partial^2 y}{\partial t^2} - cF \qquad m = J\frac{\partial^2 \vartheta}{\partial t^2} \tag{2}$$

where ρ is the (line) density of the column, ϑ is the angle between the tangent to the column axis and the $\bar{x}$-axis, J is its rotary inertia coefficient, cF is the force (per unit length) of the foundation and t is time. In what follows we shall, for simplicity, neglect the rotary inertia term, i.e., we set $J = 0$. By using (1) and (2) we obtain

$$\begin{aligned} &\frac{\partial H}{\partial S} = \rho\frac{\partial^2 x}{\partial t^2} \qquad \frac{\partial V}{\partial S} = \rho\frac{\partial^2 y}{\partial t^2} + cF \qquad \frac{\partial M}{\partial S} = -V\cos\vartheta + H\sin\vartheta \\ &\frac{\partial x}{\partial S} = \cos\vartheta \qquad \frac{\partial y}{\partial S} = \sin\vartheta \qquad \frac{\partial \vartheta}{\partial S} = \frac{M}{EI} \end{aligned} \tag{3}$$

where EI is the bending stiffness of the column. The influence of the foundation is expressed through F that we assume to be given as

$$F + \tau_F D_t^{(\alpha)} F = E_p\big(y + \tau_y D_t^{(\alpha)} y\big) \tag{4}$$

with $0 < \alpha < 1$. In (3), we use $D_t^{(\alpha)}(\cdot)$ to denote the αth derivative of a function $(\cdot)$ with respect to time taken in Riemann–Liouville form as (see [7])

$$D_t^{(\alpha)} g(t) = g^{(\alpha)} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{g(\xi)\,\mathrm{d}\xi}{(t-\xi)^\alpha} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{g(t-\xi)\,\mathrm{d}\xi}{\xi^\alpha}. \tag{5}$$

The dimension of the constants τ_y and τ_F is $[\text{time}]^\alpha$. The constants E_p, τ_F and τ_y in (3) are called instantaneous moduli of the foundation and the relaxation times, respectively. We assume that the following inequality, as a consequence of the second law of thermodynamics, is satisfied (see [8, 9])[3]

$$E > 0 \quad \tau_F > 0 \quad \tau_y > \tau_F. \tag{6}$$

Note that in the case $\alpha = 1$ the foundation becomes a standard viscoelastic solid. The boundary conditions for system (3) are

$$\begin{aligned} &y(0,t) = 0 \qquad \vartheta(0,t) = 0 \qquad H(L,t) = -F_0 \qquad M(L,t) = 0 \\ &x(0,t) = 0 \qquad y(0,t) = 0 \qquad y(L,t) = 0 \end{aligned} \tag{7}$$

corresponding to the rod shown in figure 1.

The trivial solution to the system (3), (7) in which the rod axis is straight reads

$$\begin{aligned} &H^0(S,t) = -F_0 \qquad V^0(S,t) = 0 \qquad M^0(S,t) = 0 \qquad x^0(S,t) = S \\ &y^0(S,t) = 0 \qquad \vartheta^0(S,t) = 0 \qquad F^0(S,t) = 0. \end{aligned} \tag{8}$$

The trivial solution to system (3), (7) in which the axis of the rod remains straight, reads

$$\begin{aligned} &H^0(S,t) = -F_0 \qquad V^0(S,t) = 0 \qquad M^0(S,t) = 0 \\ &x^0(S,t) = S \qquad y^0(S,t) = 0 \qquad \vartheta^0(S,t) = 0. \end{aligned} \tag{9}$$

Let $H = H^0 + \Delta H, \ldots, \vartheta = \vartheta^0 + \Delta\vartheta$. By substituting this in (3), (7) and neglecting the higher order terms in perturbations $\Delta H, \ldots, \Delta\vartheta$, we obtain

$$EI\frac{\partial^2 \Delta y}{\partial S^4} + F_0\frac{\partial^2 \Delta y}{\partial S^2} + \rho\frac{\partial^2 \Delta y}{\partial t^2} + F = 0 \tag{10}$$

subject to

$$\Delta y(0,t) = 0 \qquad \frac{\partial \Delta y(0,t)}{\partial S} = 0 \qquad \frac{\partial^2 \Delta y(L,t)}{\partial S^2} = 0 \qquad \frac{\partial^3 \Delta y(L,t)}{\partial S^3} = 0. \tag{11}$$

[3] If one uses a rheological model shown under the rod in figure 1 then the constants in (5) are given as $E = E_0E_d/(E_0 + E_d)$, $\tau_y = (\eta/E_d)^\alpha$, $\tau_F = \tau_y(E/E_0)$.

We assume the initial conditions corresponding to (10) to be known but we do not specify this now. Introducing the dimensionless quantities

$$\lambda = \frac{F_0 L^2}{EI} \qquad \tau = \frac{t}{\sqrt{\frac{\rho_0 l^4}{EI}}} \qquad u = \frac{\Delta y}{l} \qquad \xi = \frac{S}{L} \qquad \beta = c\frac{E_p L^4}{EI}$$
$$f = \frac{F}{LE_p} \qquad a = \tau_F\left(\frac{EI}{\rho L^4}\right)^{\alpha/2} \qquad b = \tau_y\left(\frac{EI}{\rho L^4}\right)^{\alpha/2} \tag{12}$$

we obtain

$$\frac{\partial^4 u}{\partial\xi^4} + \lambda\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\tau^2} + \beta f = 0 \qquad \tau > 0 \quad 0 < \xi < 1 \tag{13}$$

and

$$f + af^{(\alpha)} = u + bu^{(\alpha)} \tag{14}$$

with $0 < \alpha < 1$. The boundary conditions are

$$u(0,\tau) = 0 \qquad \frac{\partial u}{\partial\xi}(0,\tau) = 0 \qquad \frac{\partial^2 u}{\partial\xi^2}(1,\tau) = 0 \qquad \frac{\partial^3 u}{\partial\xi^3}(1,\tau) = 0 \quad \tau > 0. \tag{15}$$

The restrictions $(6)_{2,3}$ become

$$0 < a < b. \tag{16}$$

Note that for the case $a = b$ equation (14) leads to $f = u$ and the foundation becomes elastic.

3. Solution of the system (13), (14)

3.1. Separation of variables

In this section we shall analyse properties of the solution to (13)–(15). Thus we consider

$$\frac{\partial^4 u}{\partial\xi^4} + \lambda\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\tau^2} + \beta f = 0 \qquad \tau > 0 \quad 0 < \xi < 1 \qquad f + af^{(\alpha)} = u + bu^{(\alpha)} \tag{17}$$

where $b > a > 0$, $\lambda > 0$, $\beta > 0$ and $0 < \alpha < 1$, with

$$u(0,\tau) = 0 \qquad \frac{\partial u}{\partial\xi}(0,\tau) = 0 \qquad \frac{\partial^2 u}{\partial\xi^2}(1,\tau) = 0 \qquad \frac{\partial^3 u}{\partial\xi^3}(1,\tau) = 0 \quad \tau > 0. \tag{18}$$

We assume that solutions are of the form $u(\xi,\tau) = Y(\xi)T(\tau)$ and $f(\xi,\tau) = X(\xi)V(\tau)$. Then (13), (14) reduce to

$$\begin{aligned} &Y^{(4)}(\xi)T(\tau) + \lambda Y^{(2)}(\xi)T(\tau) + Y(\xi)T^{(2)}(\tau) + \beta X(\xi)V(\tau) = 0 \\ &X(\xi)V(\tau) + aX(\xi)V^{(\alpha)}(\tau) = Y(\xi)T(\tau) + bY(\xi)T^{(\alpha)}(\tau). \end{aligned} \tag{19}$$

For the existence of $(19)_2$ it is sufficient that

$$X(\xi) = AY(\xi) \qquad A(V(\tau) + aV^{(\alpha)}(\tau)) = T(\tau) + bT^{(\alpha)}(\tau) \tag{20}$$

with $A \in \mathbb{R}\backslash\{0\}$. Let us introduce in $(17)_1$ a new constant $\omega^2 \in \mathbb{R}$

$$\begin{aligned} &Y^{(4)}(\xi)T(\tau) + \lambda Y^{(2)}(\xi)T(\tau) - \omega^2 Y(\xi)T(\tau) \\ &\qquad + \omega^2 Y(\xi)T(\tau) + Y(\xi)T^{(2)}(\tau) + \beta AY(\xi)V(\tau) = 0. \end{aligned} \tag{21}$$

Thus, if we find a solution to the system

$$\begin{aligned} &X(\xi) = AY(\xi) \\ &Y^{(4)}(\xi) + \lambda Y^{(2)}(\xi) - \omega^2 Y(\xi) = 0 \\ &T^{(2)}(\tau) + \omega^2 T(\tau) + \beta AV(\tau) = 0 \\ &bT^{(\alpha)}(\tau) - aAV^{(\alpha)}(\tau) + T(\tau) - AV(\tau) = 0 \end{aligned} \tag{22}$$

we have a solution to (17) as well.

3.2. Solutions to (22)$_2$

We shall start with equation (22)$_2$. This equation with boundary conditions (18) has the same analytical form as the corresponding equation and boundary conditions for Beck's column without foundation (see [10]).

The solution to (22)$_2$ is

$$Y(\xi) = C_1 \cosh r_1\xi + C_2 \sinh r_1\xi + C_3 \cos r_2\xi + C_4 \sin r_2\xi \tag{23}$$

where C_i, $i = 1, \ldots, 4$ are constants and

$$r_1 = \sqrt{\frac{\sqrt{\lambda^2 + 4\omega^2} - \lambda}{2}} \qquad r_2 = \sqrt{\frac{\sqrt{\lambda^2 + 4\omega^2} + \lambda}{2}} \quad \omega^2 > 0. \tag{24}$$

By using (23) in boundary conditions (18) leads to the connections between λ and ω^2 given by

$$\Delta(\lambda, \omega^2) = \lambda^2 + 2\omega^2 + 2\omega^2 \cosh r_1 \cos r_2 + |\omega|\lambda \sinh r_1 \sin r_2 = 0. \tag{25}$$

When $\lambda = 0$ we have infinite number of positive real values of ω^2 laying in pairs between $\left(\frac{\pi}{2} + 2k\pi\right)^2$ and $\left(\frac{3\pi}{2} + 2k\pi\right)^2$, $k = 0, 1, \ldots$ and satisfying $\Delta(\lambda, \omega^2) = 0$ (cf [11]). In the applications we are interested in the lowest values of λ for which the rod loses stability. Thus, we consider the first two values of ω^2. If λ is increased, then the two corresponding values of ω^2 approach each other until they meet at a certain value $\lambda = \lambda_{cr}$ ($\lambda \approx 20.05$, the critical value).

As λ is increased beyond λ_{cr}, the value of ω^2 becomes complex, with real part equal to $\omega^2 = \omega^2_{cr} \approx 121.25$ (cf [12]). More precisely, if $\lambda \in (0, \lambda_{cr} + \varepsilon)$, with $\varepsilon > 0$ and small enough, then the values of ω^2 determined from (25) are either positive numbers or complex numbers with the real parts close to ω^2_{cr}.

Since the case $\omega^2 > 0$ has been treated, we consider (23) only for the case of ω^2 complex. Let $\omega^2 = \omega^2_{cr} + \mathrm{i}q$; $q \in \mathbb{R}$, $q \neq 0$. The characteristic equation corresponding to (22)$_2$ reads

$$r^4 - \lambda r^2 - \omega^2 = 0.$$

This equation has four roots

$$r_{1,2,3,4} = \pm\frac{1}{\sqrt{2}}\sqrt{-\lambda \pm \sqrt{\lambda^2 + 4\omega^2}}. \tag{26}$$

Let us remark that we have to repeat the operation of the extraction of the square root of complex numbers twice. We analyse this operation. Let $c + \mathrm{i}d$ be a complex number. Then

$$\sqrt{c + \mathrm{i}d} = x_j + \mathrm{i}y_j \qquad j = 1, 2 \tag{27}$$

where

$$x_{1,2} = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{c^2 + d^2} + c} \qquad y_{1,2} = \pm\frac{1}{\sqrt{2}}\frac{d}{\sqrt{d^2}}\sqrt{\sqrt{c^2 + d^2} - c}.$$

Remark. By combining (26) and (27) we conclude that there is always a root given by (26) with a positive real part.

3.3. Solutions to system (22)$_3$, (22)$_4$

Applying the Laplace transform ($(\mathcal{L}(f)(s) = \hat{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t)$) to the system (22)$_3$, (22)$_4$ we have

$$\begin{aligned}(s^2+\omega^2)\hat{T}(s) + \beta A \hat{V}(s) &= sT_0 + T_1\\ (bs^\alpha + 1)\hat{T}(s) - (aAs^\alpha + A)\hat{V}(s) &= 0\end{aligned} \tag{28}$$

where $T_0 = T(0)$ and $T_1 = T^{(1)}(0)$.

Let $\Delta_0(s)$, $\Delta_1(s)$ and $\Delta_2(s)$ denote

$$\begin{aligned}\Delta_0(s) &= \begin{vmatrix} s^2+\omega^2 & \beta A\\ bs^\alpha+1 & -A(as^\alpha+1)\end{vmatrix}\\
&= -Aa\left[\left(s^\alpha+\frac{1}{a}\right)\left(s^2+\frac{a\omega^2+b\beta}{a}\right)+\frac{1}{a}\left(1-\frac{b}{a}\beta\right)\right]\\
&= -A[as^{\alpha+2}+s^2+(a\omega^2+\beta b)s^\alpha+\omega^2+\beta]\\
\Delta_1(s) &= \begin{vmatrix} sT_0+T_1 & \beta A\\ 0 & -A(as^\alpha+1)\end{vmatrix}\\
&= -Aa(T_0 s+T_1)\left(s^\alpha+\frac{1}{a}\right)\\
\Delta_2(s) &= \begin{vmatrix} s^2+\omega^2 & sT_0+T_1\\ bs^\alpha+1 & 0\end{vmatrix}\\
&= -b(T_0 s+T_1)\left(s^\alpha+\frac{1}{b}\right)\\
&= -b(T_0 s+T_1)\left(s^\alpha+\frac{1}{a}\right)-\left(1-\frac{b}{a}\right)(T_0 s+T_1).\end{aligned} \tag{29}$$

The solution to the system (28) is

$$\begin{aligned}\hat{T}(s) &= \frac{\Delta_1(s)}{\Delta_0(s)} = \frac{\left(s^\alpha+\frac{1}{a}\right)(T_0 s+T_1)}{\left(s^\alpha+\frac{1}{a}\right)\left(s^2+\frac{a\omega^2+b\beta}{a}\right)+\frac{1}{a}\left(1-\frac{b}{a}\beta\right)}\\
\hat{V}(s) &= \frac{\Delta_2(s)}{\Delta_0(s)} = \frac{b\left(s^\alpha+\frac{1}{a}\right)(T_0 s+T_1)+\left(1-\frac{b}{a}\right)(T_0 s+T_1)}{Aa\left[\left(s^\alpha+\frac{1}{a}\right)\left(s^2+\frac{a\omega^2+b\beta}{a}\right)+\frac{1}{a}\left(1-\frac{b}{a}\beta\right)\right]}.\end{aligned} \tag{30}$$

We introduce new notation $p = \frac{a\omega^2+b\beta}{a}$ and $\frac{1}{a}\left(1-\frac{b}{a}\beta\right) = c$ and we consider the expression $\frac{-Aa}{\Delta_0(s)}$. Let us remark that c is a real number and p can be a positive real number or a complex number $\left(p = \omega_{\mathrm{cr}}^2 + \frac{\beta b}{a} + \mathrm{i}q\right)$ which depends on ω^2,

$$\begin{aligned}\frac{-Aa}{\Delta_0(s)} &= \frac{1}{\left(s^\alpha+\frac{1}{a}\right)(s^2+p)+c}\\
&= \frac{1}{\left(s^\alpha+\frac{1}{a}\right)(s^2+p)}\left(1+\sum_{\nu=1}^{\infty}(-c)^\nu\left(\frac{1}{s^\alpha+\frac{1}{a}}\right)^\nu\left(\frac{1}{s^2+p}\right)^\nu\right).\end{aligned} \tag{31}$$

We have proved (cf [19]) that there is a continuous function $\phi(t)$, $t \geqslant 0$ such that

$$(\mathcal{L}\phi)(s) = \sum_{\nu=1}^{\infty}(-c)^\nu\left(\frac{1}{s^\alpha+\frac{1}{a}}\right)^\nu\left(\frac{1}{s^2+p}\right)^\nu \tag{32}$$

and

$$\phi(t) = \sum_{\nu=1}^{\infty}(-c)^\nu\left(\mathcal{L}^{-1}\left(\frac{1}{s^2+p}\right)*w\right)^{*\nu}(t) \tag{33}$$

where '$*$' denotes convolution ($((f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau)$ and $F^{*\nu}$ means ν-fold convolution of F. Also $\mathcal{L}^{-1}\big(\frac{1}{s^2+p}\big)$ one can find in any tables for the Laplace transform (cf (37)) and

$$w(t) = \mathcal{L}^{-1}\left(\frac{1}{s^\alpha + \frac{1}{a}}\right)(t) = \alpha t^{\alpha-1}E_\alpha^{(1)}(z) \qquad z = -\left(\frac{1}{a}t^\alpha\right) \qquad t \geqslant 0 \tag{34}$$

where $E_\alpha(z)$ is the Mittag–Leffler function (cf [15]).

By (28) and (30) one can see that two functions $\hat{f}(s) = \big(s^\alpha + \frac{1}{a}\big)\frac{-Aa}{\Delta_0(s)}$ and $s\hat{f}(s)$ have the basic role in the process of determining $T(t)$ and $V(t)$. Therefore we first consider these two functions. Thus,

$$\hat{f}(s) = \left(s^\alpha + \frac{1}{a}\right)\frac{-Aa}{\Delta_0(s)} = \frac{1}{s^2+p}(1+\hat{\phi}(s)) \tag{35}$$

and

$$f(t) = \mathcal{L}^{-1}(\hat{f}(s))(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+p}\right)(t) + \left(\mathcal{L}^{-1}\left(\frac{1}{s^2+p}\right) * \phi\right)(t) \tag{36}$$

where

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+p}\right)(t) = \frac{1}{p}\frac{\mathrm{e}^{\mathrm{i}\sqrt{p}t} - \mathrm{e}^{\mathrm{i}\sqrt{p}t}}{2\mathrm{i}}. \tag{37}$$

In all three cases $f(0) = 0$. Consequently $s\hat{f}(s) = \mathcal{L}(f^{(1)})(s)$, i.e.

$$f^{(1)}(t) = \mathcal{L}^{-1}\left(\frac{s}{s^2+p}\right)(t) + \left(\mathcal{L}^{-1}\left(\frac{s}{s^2+p}\right) * \phi\right)(t) \tag{38}$$

where

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+p}\right) = \frac{1}{\sqrt{p}}\frac{\mathrm{e}^{\mathrm{i}\sqrt{p}t} + \mathrm{e}^{\mathrm{i}\sqrt{p}t}}{2}. \tag{39}$$

The solution to $(22)_3$, $(22)_4$ has the form

$$\begin{aligned} T(t) &= T_0 f^{(1)}(t) + T_1 f(t) \\ V(t) &= \frac{b}{aA}T_0 f^{(1)} + \frac{b}{aA}T_1 f(t) + \frac{1}{aA}\left(1 - \frac{b}{a}\right)[T_0(w * f^{(1)})(t) + T_1(w * f)(t)] \end{aligned} \tag{40}$$

where f, $f^{(1)}$ and w have been given by (36), (38) and (34), respectively.

4. Properties of the solution to $(22)_3$, $(22)_4$

4.1. Solution (40) is a classical one

We shall first prove that $f \in C^3([0,\infty))$. By (38) it follows that $f \in C^1([0,\infty))$. Let us consider $f^{(2)}(t)$, $t \geqslant 0$. Since $\mathcal{L}^{-1}\big(\frac{s}{s^2+p}\big)(t)$ is a smooth function, we have to analyse only

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{L}^{-1}\left(\frac{s}{s^2+p}\right) * \phi\right)(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathcal{L}^{-1}\left(\frac{s}{s^2+p}\right)(t-\tau)\phi(\tau)\,\mathrm{d}\tau \\ &= \left(\mathcal{L}^{-1}\left(\frac{s}{s^2+p}\right)(0)\phi(t)\right) + \int_0^t \left(\mathcal{L}^{-1}\left(\frac{s}{s^2+p}\right)\right)_t^{(1)}(t-\tau)\phi(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{41}$$

We know that $w(t)$ is a smooth function on $(0,\infty)$ and $w(t) = O(t^{\alpha-1})$, $t \to 0$. Then $\phi(t)$ given by (33) has the property that $|\phi^{(i)}(t)| \leqslant O(t^\alpha)$, $t \to 0$, $i = 0, 1$. Consequently, (41) gives $f \in C^3([0,\infty))$.

Now it is easily seen that $T \in C^2([0,\infty))$ and that $V^{(\alpha)}(t) \in C^0([0,\infty))$. Solution (40) is a classical solution. Also $T^{(i)}$, $i = 0, 1, 2$, and V, $V^{(\alpha)}$ are J functions (cf [14]). For the definition of J functions see [14], volume I, p 29. For the J functions there exists the Laplace transform. This shows that the use of the Laplace transform was correct in the present situation.

4.2. *Asymptotic behaviour of the solution* $(22)_3$, $(22)_4$

In section 3.2 we have seen that if the solution $Y(\xi)$, given by (23), satisfies boundary conditions (18), then ω^2 is a positive real number or a complex number. Therefore, we discuss the asymptotic behaviour of the functions $T(t)$ and $V(t)$ in (40) only for these values of ω^2. We know that this asymptotic behaviour depends on the real parts of zeros of $\Delta_0(s)$. But for the discussion of asymptotic behaviour we can use the analytical form of $T(t)$ and $V(t)$ too. We shall combine these two possibilities.

Let us start with the case $\omega^2 > 0$. Many authors studied zeros of a complex function using different approaches (see [16–18]). In [19] we used quite elementary analysis. These results are applicable to (40) as well. They give for (40) the following.

Let us consider

$$-\frac{1}{A}\Delta_0(s) = as^{\alpha+2} + s^2 + (a\omega^2 + \beta b)s^\alpha + \omega^2 + \beta. \tag{42}$$

The coefficient $a\omega^2 + b\beta$ can be written in the form

$$a\omega^2 + b\beta = a(\omega^2 + \beta) + \beta(b - a). \tag{43}$$

Since $\beta > 0$ and $b > a$, it follows that $a\omega^2 + b\beta > 0$. Also by (43) we have

$$\frac{a(\omega^2 + \beta) + \beta(b - a)}{a} = \omega^2 + \beta.$$

In [18], p 518 it was proved that in this case $\Delta_0(s)$ has no zeros neither real and positive nor complex with positive real part. Consequently, if $\omega^2 > 0$, then the solutions given by (40) are stable. We can arrive at the same conclusion by (37), (39) because in this case $p > 0$.

We shall now discuss the asymptotic behaviour of solutions (40) in the case when ω^2 is a complex number, $\omega^2 = \omega_{\mathrm{cr}}^2 + \mathrm{i}q$, $q \in \mathbb{R}$, $q \neq 0$. We first transform (43) to another form, so that

$$s^{\alpha+2} + \frac{1}{a}s^2 + \left(\omega^2 + \beta\frac{b}{a}\right)s^\alpha + \frac{\omega^2 + \beta}{a} = 0$$

is equivalent to

$$\left(s^2 + \omega^2 + \beta\frac{b}{a}\right)\left(s^\alpha + \frac{1}{a}\right) = \frac{\beta}{a^2}(b - a). \tag{44}$$

If there existed a $\rho_0 \geqslant 0$ such that

$$\left(\rho_0^2 + \omega^2 + \beta\frac{b}{a}\right)\left(\rho_0^\alpha + \frac{1}{a}\right) = \frac{\beta}{a^2}(b - a) \tag{45}$$

(ρ_0^α is the principal branch), then we would have that the product in (45) is a positive real number, but one factor is a complex number (which is not real) and other factor is a real number. Since this is not possible, it follows that there is no solution $\rho_0 \geqslant 0$ of (44) and solutions (40) cannot exponentially diverge. This follows also from (37), (39) as well, because p cannot be negative.

By (27) for $\omega^2 = \omega_{\mathrm{cr}}^2 + \mathrm{i}q$ we have

$$\sqrt{p} = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\omega_{\mathrm{cr}}^2 + \beta\frac{b}{a}\right)^2 + q^2} + \left(\omega_{\mathrm{cr}}^2 + \beta\frac{b}{a}\right)}$$
$$\pm\frac{\mathrm{i}}{\sqrt{2}}\frac{q}{\sqrt{q^2}}\sqrt{\sqrt{\left(\omega_{cr}^2 + \beta\frac{b}{a}\right)^2 + q^2} - \left(\omega_{\mathrm{cr}}^2 + \beta\frac{b}{a}\right)}.$$

By using (37), (39) we conclude that in this case we have oscillations with increasing amplitude, i.e. a flutter (cf remark in section 3.2).

4.3. Approximation of series (33)

There are different inversion formulae for the Laplace transform (cf [14]). We made good use of the series (32). Consequently in solution (40) we have series (33). We shall estimate the error when we take the sum

$$\sum_{\nu=1}^{m}(-c)^\nu\left(\mathcal{L}^{-1}\left(\frac{1}{s^2+p}\right)*w\right)^{*\nu}(t) \qquad 0\leqslant t\leqslant t_0$$

instead of series (33) but only if $p>0$, because if we have the stability of solution to (40), then $p>0$, $p=(a\omega^2+b\beta)/a=\omega^2+\beta+\beta\left(\frac{b}{a}-1\right)$, i.e. $\omega^2>0$. It is easily seen (cf [19], p 511) that for any $\nu\in\mathbb{N}$ and $0\leqslant t\leqslant t_0$, $p>0$

$$\left|\left(\mathcal{L}^{-1}\left(\frac{1}{s^2+p}\right)*w\right)^{*\nu}(t)\right|\leqslant C^\nu(t_0)\frac{t^{(\alpha+1)\nu-1}}{\Gamma(\nu(\alpha+1))}$$

where $\Gamma(x)$ is the Euler gamma function. Then with $C_0=cC(t_0)$ we have

$$\begin{aligned}
&\left|\sum_{\nu=m+1}^{\infty}(-c)^\nu\left(\mathcal{L}^{-1}\left(\frac{1}{s^2+p}\right)*w\right)^{*\nu}(t)\right|\\
&\qquad\leqslant\sum_{\nu=m+1}^{\infty}C_0^\nu\frac{t^{(\alpha+1)\nu-1}}{\Gamma(\nu(\alpha+1))}\\
&\qquad\leqslant\sum_{k=1}^{\infty}\frac{C_0^{k+m}t^{(\alpha+1)(k+m)-1}}{\Gamma(k(\alpha+1)+m(\alpha+1))}\\
&\qquad\leqslant C_0^m t^{m(\alpha+1)-1}\sum_{k=1}^{\infty}\frac{(C_0t^{\alpha+1})^k}{\Gamma(k(\alpha+1)+m(\alpha+1))}\\
&\qquad\leqslant C_0^m t^{m(\alpha+1)-1}C_0t^{\alpha+1}\sum_{n=0}^{\infty}\frac{(C_0t^{\alpha+1})^n}{\Gamma(n(\alpha+1)+(m+1)(\alpha+1))}\\
&\qquad\leqslant C_0^{m+1}t^{(m+1)(\alpha+1)-1}\sum_{n=0}^{\infty}\frac{(C_0t^{\alpha+1})^n}{\Gamma(n(\alpha+1)+(m+1)(\alpha+1))}\\
&\qquad\leqslant C_0^{m+1}t^{(m+1)(\alpha+1)-1}E_{\alpha+1,(m+1)(\alpha+1)}\left(C_0t^{\alpha+1}\right)
\end{aligned}\qquad(46)$$

where $E_{u,v}$ is a function similar to Mittag–Leffler's function (cf [15], p 210). Properties of the function $E_{u,v}(z)$, which can be useful in our case one can find in [15], p 210.

Evidently estimate (46) is interesting before all in the neighbourhood of $t=0$.

5. Interpretation of the results and conclusion

We summarize now the results obtained in this paper in the context of a physical model, i.e., Beck's column on a viscoelastic foundation.

(1) The time evolution of the system (13)–(15) is determined by the solution of the following system of differential equations containing fractional derivatives (see $(22)_{2,3}$),

$$\begin{aligned} &T^{(2)}(\tau) + \omega^2 T(\tau) + \beta A V(\tau) = 0 \\ &bT^{(\alpha)}(\tau) - aAV^{(\alpha)}(\tau) + T(\tau) - AV(\tau) = 0 \end{aligned} \tag{47}$$

where $0 < a < b, \beta > 0$ and $A \in \mathbb{R}\backslash\{0\}$. For (47) the initial conditions should be prescribed and we consider them arbitrary. For the case of a column on a viscoelastic foundation, as for the case of a column without foundation, for given λ in (13) the corresponding ω^2 is determined from (25). This relation leads to $\omega^2 > 0$ if $0 \leqslant \lambda \leqslant 20.0509$ and $\omega^2 = \omega^2_{\mathrm{cr}} + \mathrm{i}q$, with $\omega^2_{\mathrm{cr}} \approx 121.25$ (cf [12]) if $\lambda \in (20.0599, 20.0599 + \varepsilon)$ with $\varepsilon > 0$ small enough.

In the first case, that is for $0 \leqslant \lambda \leqslant 20.0509$, we have stability of the rod, i.e., the solution to (47) is bounded. In the second case the solution to (47) becomes unstable with both $T(t)$ and $V(t)$ representing oscillations with increasing amplitude (there is no exponential type of instability).

(2) The stability results just stated show that the viscoelastic foundation of fractional derivative type, for Beck's column, does not increase the stability bound. This result, often called the Hermann–Smith paradox, was known to hold for the Winkler type of foundation (see [1]). Here we show that it holds for a viscoelastic foundation of fractional derivative type too.

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